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BOUNDARY-VALUE PROBLEM OF DYNAMIC GEOMETRICALLY NONLINEAR ELASTICITY

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The linear theory of elasticity does not provide the accuracy necessary in a number of important elastic problems, and one of the existing nonlinear theories is used in its place. The nonlinearity of these theories is connected not only with the law that governs the mechanical behavior of the material (physical nonlinearity), but also with the dependence of the strains on the gradients of the displacements (geometric nonlinearity). Here, we examine a two-dimensional dynamic problem in the Novozhilov variant of geometrically nonlinear elasticity. We derive equations in stresses and rotations, represent these quantities in terms of potentials, and construct equations for the potentials.

We show that there is an interaction between expansion–compression and shear waves in the material when allowance is made for nonlinearity. We identify a class of solutions to the equations of motion that contains two arbitrary functions and show its application to the solution of the boundary-value problem of the stress distribution in a semi-infinite elastic medium during the motion of a pressure pulse along its surface.

The nonlinear model of elasticity proposed by V. V. Novozhilov [1] is described by equations of motion, Hooke's law, a special nonlinear relation linking strains with extensions and rotations, and equations expressing the latter in terms of displacements:

$$\begin{gathered}\rho(\mathbf{f}-\mathbf{u}_{,tt})+\operatorname{div}P=0,\ \rho=\text{const},\\ P=\lambda\varepsilon_1 G+2\mu\varepsilon,\ \ \varepsilon_1=\operatorname{tr}\varepsilon,\\ 2\varepsilon=2e+\omega\cdot\omega,\ \ 2e=\nabla\mathbf{u}+\mathbf{u}\nabla,\ \ 2\omega=\nabla\mathbf{u}-\mathbf{u}\nabla.\end{gathered}\tag{1}$$

Here, $\mathbf{u}$ and $\mathbf{f}$ are the displacement vectors and the densities of the body forces; G, P, ε, e, ω, $\nabla\mathbf{u}$, and $\mathbf{u}\nabla$ are the metric tensor and the tensors of the stresses, strains, extensions, and rotations, the displacement gradient, and the transposed displacement gradient; ρ, λ, and μ are the density of the material and the Lamé constants.

Model (1) was obtained in the long-wave approximation with the assumption that the small rotations occurring in the material may be considerably greater than its small extensions. Thus, the squares of the former will be comparable in magnitude to the latter. Such a situation might be realized, for example, in flexible bodies or in bodies having cavities near their internal and external boundaries.

System (1) generalizes the dynamic equations of the linear theory of elasticity, differing from these equations only in the presence of the nonlinear term in the representation of the strains in terms of extensions and shears. In the case of a two-dimensional problem, Eqs. (1) appear as follows in the complex coordinates corresponding to the actual state $z = x + iy$, $\bar{z} = x - iy$ (x, y are cartesian coordinates)

$$\begin{gathered}\rho\left(f-\frac{\partial^2 u}{\partial t^2}\right)+\frac{\partial P^{11}}{\partial z}+\frac{\partial P^{12}}{\partial \bar{z}}=0,\\ P^{11}=\overline{P^{22}}=2\mu\varepsilon^{11},\ \ P^{21}=P^{12}=2(\lambda+\mu)\varepsilon^{21},\\ \varepsilon^{11}=\overline{\varepsilon^{22}}=e^{11},\ \ \varepsilon^{21}=\varepsilon^{12}=e^{21}+\frac{1}{4}(\omega^{21})^2,\\ e^{11}=\overline{e^{22}}=2\frac{\partial u}{\partial \bar{z}},\ \ e^{21}=e^{12}=\frac{\partial u}{\partial z}+\frac{\partial \bar{u}}{\partial \bar{z}},\ \ \omega^{21}=\overline{\omega^{12}}=\frac{\partial u}{\partial z}-\frac{\partial \bar{u}}{\partial \bar{z}}.\end{gathered}\tag{2}$$

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The complex contravariant components of the vectors and tensors in (2) are related to the cartesian components of the corresponding quantities (designated by the same symbols as above, but with letter subscripts) by the formulas [2]

$$\begin{gathered} u = u^1 = u_x + iu_y,\ \bar{u} = u^2 = u_x - iu_y, \\ P^{11} = P_{xx} - P_{yy} + 2iP_{xy},\ P^{21} = P_{xx} + P_{yy}, \\ \omega^{11} = 0,\ \omega^{22} = 0,\ \omega^{21} = 2i\omega_{xy}. \end{gathered} \tag{3}$$

We can use (2) to represent the displacement gradients through the stresses and rotations:

$$4\mu \frac{\partial u}{\partial z} = \frac{\mu}{\lambda + \mu} P^{21} + 2\mu\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right),\ 4\mu \frac{\partial u}{\partial \bar{z}} = P^{11}. \tag{4}$$

The compatibility condition for these equalities gives us the compability equation for the stresses and rotation. This equation, together with dynamic equation (2), forms a system of two complex equations for the complex and real stresses P^{11}, P^{21} and the purely imaginary rotation ω^{21}:

$$\frac{\partial P^{11}}{\partial z} = \frac{\partial}{\partial \bar{z}}\left[\frac{\mu}{\lambda + \mu} P^{21} + 2\mu\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right]; \tag{5}$$

$$\left(\Delta - \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right) P^{11} + 4\frac{\partial^2 P^{21}}{\partial \bar{z}^2} + 4\rho\frac{\partial f}{\partial \bar{z}} = 0,\ \Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} \tag{6}$$

(Δ is the Laplace operator).

Equations (4)-(6) allow the stresses, rotations, and displacements to be represented through the complex potential function $J(z, \bar{z}) = J_1 + iJ_2$. Equation (5) will in fact be satisfied if we put

$$P^{11} = 4\frac{\partial^2 J}{\partial \bar{z}^2},\ \frac{\mu}{\lambda + \mu} P^{21} + 2\mu\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right) = 4\frac{\partial^2 J}{\partial z \partial \bar{z}}.$$

We use this equation and (4) to express the stresses, rotations, and displacements in terms of potentials:

$$P^{11} = 4\frac{\partial^2 (J_1 + iJ_2)}{\partial \bar{z}^2},\ P^{21} = \frac{\lambda + \mu}{\mu}\left(\Delta J_1 - \frac{1}{8\mu}(\Delta J_2)^2\right); \tag{7}$$

$$\omega^{21} = \frac{i}{2\mu}\Delta J_2,\ \mu u = \frac{\partial J_1}{\partial \bar{z}} + i\frac{\partial J_2}{\partial \bar{z}}. \tag{8}$$

Having expressed the body force through the potential $\Psi = \Psi_1 + i\Psi_2$ by means of the relation $\rho f = 2\partial\Psi/\partial\bar{z}$ and having expressed the stresses with Eqs. (7), we represent (6) in the form of the equation for the potentials:

$$4\frac{\partial^2}{\partial \bar{z}^2}\left\{\left(\Delta - \frac{\rho}{\mu}\frac{\partial^2}{\partial t^2}\right)(J_1 + iJ_2) + \frac{\lambda + \mu}{\mu}\left[\Delta J_1 - \frac{1}{8\mu}(\Delta J_2)^2\right] + 2(\Psi_1 + i\Psi_2)\right\} = 0.$$

This equation is identically satisfied if the complex expression in the braces is equal to zero, i.e. if its real and imaginary parts vanish:

$$\begin{gathered} \left(\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) J_1 - \frac{1}{8\mu}\frac{c_1^2 - c_2^2}{c_1^2}(\Delta J_2)^2 + 2\frac{c_2^2}{c_1^2}\Psi_1 = 0, \\ \left(\Delta - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right) J_2 + 2\Psi_2 = 0,\ c_1^2 = \frac{\lambda + 2\mu}{\rho},\ c_2^2 = \frac{\mu}{\rho}. \end{gathered} \tag{9}$$

Thus, for potentials satisfying Eqs. (9), Eqs. (7) and (8) give the solution of the equations of motion.

We can conclude from displacement equations (8) and potential equations (9) that the potential J_1 defines an expansion–compression wave propagating at the velocity c_1, while the potential J_2 defines a shear wave propagating at the velocity c_2.

Equalities (9) also show that the shear wave behaves in the same manner in the nonlinear medium as does a linear wave. The shear wave acts on the expansion–compression wave as a body force, which shows that there is some interaction between the waves in a nonlinear medium [3].

Let us now examine the class of problems concerning the motion of a semi-infinite medium $y \geq 0$ without body forces. Perturbations are propagated along the surface of the medium at a constant velocity c parallel to the x-axis. In this case, a steady state will exist in the corresponding coordinate system ξ, η, where $\xi = x - ct$, $\eta = y$. As a result, we can take $J(x, y, t) = J(\xi, \eta)$. Accordingly, instead of the complex variables z, $\bar{z}$, t, we need only examine the complex coordinates z_1, $\bar{z}_1$ or z_2, $\bar{z}_2$. These coordinates are related to the previous complex variables and to one another by the relations

$$
\begin{gathered}
z_1 = \xi + i\beta_1\eta = \frac{1+\beta_1}{2}z + \frac{1-\beta_1}{2}\bar{z} - ct,\ \beta_1^2 = 1 - \frac{c^2}{c_1^2},\\
z_2 = \xi + i\beta_2\eta = \frac{1+\beta_2}{2}z + \frac{1-\beta_2}{2}\bar{z} - ct,\ \beta_2^2 = 1 - \frac{c^2}{c_2^2},\\
z_2 = \frac{\beta_1+\beta_2}{2\beta_1}z_1 + \frac{\beta_1-\beta_2}{2\beta_1}\bar{z}_1,\ z_1 = \frac{\beta_2+\beta_1}{2\beta_2}z_2 + \frac{\beta_2-\beta_1}{2\beta_2}\bar{z}_2.
\end{gathered}
\tag{10}
$$

In the variables (10), the differential operators figuring into (7)-(9) have the form

$$
\begin{aligned}
\Delta &= (1-\beta_1^2)\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial \bar{z}_1^2}\right) + 2(1+\beta_1^2)\frac{\partial^2}{\partial z_1\partial\bar{z}_1}\\
&= (1-\beta_2^2)\left(\frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial \bar{z}_2^2}\right) + 2(1+\beta_2^2)\frac{\partial^2}{\partial z_2\partial\bar{z}_2},\\
4\frac{\partial^2}{\partial \bar{z}^2} &= (1-\beta_1)^2\frac{\partial^2}{\partial z_1^2} + (1+\beta_1)^2\frac{\partial^2}{\partial \bar{z}_1^2} + 2(1-\beta_1^2)\frac{\partial^2}{\partial z_1\partial\bar{z}_1}\\
&= (1-\beta_2)^2\frac{\partial^2}{\partial z_2^2} + (1+\beta_2)^2\frac{\partial^2}{\partial \bar{z}_2^2} + 2(1-\beta_2^2)\frac{\partial^2}{\partial z_2\partial\bar{z}_2},\\
\Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} &= 4\beta_1^2\frac{\partial^2}{\partial z_1\partial\bar{z}_1} = (\beta_1^2-\beta_2^2)\left(\frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial \bar{z}_2^2}\right) + 2(\beta_1^2+\beta_2^2)\frac{\partial^2}{\partial z_2\partial\bar{z}_2},\\
\Delta - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} &= 4\beta_2^2\frac{\partial^2}{\partial z_2\partial\bar{z}_2} = (\beta_2^2-\beta_1^2)\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial \bar{z}_1^2}\right) + 2(\beta_2^2+\beta_1^2)\frac{\partial^2}{\partial z_1\partial\bar{z}_1},\\
2\frac{\partial}{\partial \bar{z}} &= (1-\beta_1)\frac{\partial}{\partial z_1} + (1+\beta_1)\frac{\partial}{\partial \bar{z}_1} = (1-\beta_2)\frac{\partial}{\partial z_2} + (1+\beta_2)\frac{\partial}{\partial \bar{z}_2},
\end{aligned}
\tag{11}
$$

so that the potential equations (9) (with $\Psi_1 = \Psi_2 = 0$) take the form of the Laplace and Poisson equations

$$
\frac{\partial^2 J_2}{\partial z_2\partial\bar{z}_2} = 0,\ \frac{\partial^2 J_1}{\partial z_1\partial\bar{z}_1} = \frac{w}{\mu},\ w = \frac{1}{32\beta_1^2}\frac{\beta_1^2-\beta_2^2}{1-\beta_1^2}(\Delta J_2)^2.
\tag{12}
$$

The below formulas give the general solutions of Eqs. (12) in terms of arbitrary analytic functions $F_1(z_1)$ and $F_2(z_2)$ and the particular solution $F(z_1, \bar{z}_1)$ of the second of these functions

$$
\begin{aligned}
J_2(z_2,\bar{z}_2) &= \frac{1+\beta_2^2}{2i\beta_2}[F_2(z_2) - \overline{F_2(z_2)}],\\
J_1(z_1,\bar{z}_1) &= \frac{1}{\mu}F(z_1,\bar{z}_1) - F_1(z_1) - \overline{F_1(z_1)}.
\end{aligned}
\tag{13}
$$

Assuming that at $|z_1| \to \infty$ $|w| = O(1/|z_1|^{1+\alpha})$, $\alpha > 0$, and allowing for the expressions of the function J_2 and operators (11), we can represent the particular solution in the form of an integral over the region S occupied by the elastic body [4]:

$$
F(z_1,\bar{z}_1) = \frac{2}{\pi}\int\int_S w(\xi',\beta_1\eta')\ln|z'_1 - z_1|d\xi' d(\beta_1\eta'),\ z'_1 = \xi' + i\beta_1\eta',
$$

$$w = \tau [\mathrm{Im} F''_2(z_2)]^2 = \tau [\mathrm{Im} F''_2(z_1, \bar{z}_1)]^2, \ \tau = (1 - \beta_2^2)\frac{\beta_1^4 - \beta_2^4}{2}\left(\frac{1 + \beta_2^2}{4\beta_1\beta_2}\right). \tag{14}$$

By virtue of (7)-(8) and (11), potentials (13) correspond to the following complex stresses, rotation, and displacements:

$$\begin{aligned}
P^{21} &= -2\mathrm{Re}\Big\{(\beta_1^2 - \beta_2^2)[F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1}]\Big\} + (\beta_1^2 + \beta_2^2)\frac{\tau}{\mu}[\mathrm{Im} F''_2(z_2)]^2,\\
P^{11} &= -2\mathrm{Re}\Big\{(1 + \beta_1^2)[F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1}]\\
&\quad + (1 + \beta_2^2)F''_2(z_2) - (1 - \beta_1^2)\frac{\tau}{\mu}[\mathrm{Im} F''_2(z_2)]^2\Big\}\\
&\quad + 4i\mathrm{Im}\Big\{\beta_1[F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1}] + \frac{(1 + \beta_2^2)^2}{4\beta_2} F''_2(z_2)\Big\},\\
\mu\omega^{21} &= i\frac{1 - \beta_2^4}{2\beta_2}\mathrm{Im}\, F''_2(z_2),\\
\mu u &= -\mathrm{Re}\Big\{F'_1(z_1) - \frac{1}{\mu} F_{z_1} + \frac{1 + \beta_2^2}{2} F'_2(z_2)\Big\}\\
&\quad + i\mathrm{Im}\Big\{\beta_1[F'_1(z_1) - \frac{1}{\mu} F_{z_1}] + \frac{1 + \beta_2^2}{2\beta_2} F'_2(z_2)\Big\}.
\end{aligned} \tag{15}$$

Here, in accordance with (14), the derivatives of the particular solution have the form

$$F_{z_1} = -\frac{1}{\pi}\iint_S \frac{w(\xi', \beta_1\eta')}{z'_1 - z_1} d\xi' d(\beta_1\eta'), \ F_{z_1 z_1} = -\frac{1}{\pi}\iint_S \frac{w(\xi', \beta_1\eta')}{(z'_1 - z_1)^2} d\xi' d(\beta_1\eta')$$

with the integral in the second equation being regarded as the Cauchy principle value. We can obtain expressions for the cartesian components of the quantities by separating the real and imaginary parts of Eqs. (3) and (15) and solving the resulting system of equations for the stresses. This gives us

$$\begin{aligned}
P_{xx} &= -\mathrm{Re}\Big\{(1 + 2\beta_1^2 - \beta_2^2)[F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1}]\\
&\quad + (1 + \beta_2^2)F''_2(z_2)\Big\} + (1 - 2\beta_1^2 - \beta_2^2)\frac{\tau}{\mu}[\mathrm{Im}\, F''_2(z_2)]^2,\\
P_{yy} &= (1 + \beta_2^2)\mathrm{Re}\,\{F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1} + F''_2(z_2)\} - (1 + \beta_2^2)\frac{\tau}{\mu}[\mathrm{Im}\, F''_2(z_2)]^2,\\
P_{xy} &= \mathrm{Im}\,\{2\beta_1[F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1}] + \frac{(1 + \beta_2^2)^2}{2\beta_2} F''_2(z_2)\},\\
\mu\omega_{xy} &= \frac{1 - \beta_2^4}{4\beta_2}\mathrm{Im}\, F''_2(z_2),\\
\mu u_x &= -\mathrm{Re}\,\{F'_1(z_1) - \frac{1}{\mu} F_{z_1} + \frac{1 + \beta_2^2}{2} F'_2(z_2)\},\\
\mu u_y &= \mathrm{Im}\{\beta_1[F'_1(z_1) - \frac{1}{\mu} F_{z_1}] + \frac{1 + \beta_2^2}{2\beta_2} F'_2(z_2)\}.
\end{aligned} \tag{16}$$

These formulas establish a class of exact solutions of nonlinear dynamic equations of elasticity that depends on two arbitrary analytic functions $F_1(z_1)$ and $F_2(z_2)$. These functions are found from the boundary conditions.

As follows from (16), the generalized displacements and rotation ($U_x = \mu u_x$, $U_y = \mu u_y$, $\Omega_{xy} = \mu\omega_{xy}$), as the stresses, are determined through the above potentials and are thus finite. Let L_0 and P_0 be the characteristic length and characteristic stress and let $\sigma = P_0/\mu$ be the characteristic dimensionless stress. We relate the quantities being examined to the corresponding dimensionless quantities (denoted by asterisks) on the basis of the formulas

$$z_1 = L_0 z_1^*, \ U_x = L_0 P_0 U_x^*, \ P_{xx} = P_0 P_{xx}^*, \ \Omega_{xy} = P_0 \Omega_{xy}^*,$$
$$F'_1 = L_0 P_0 F'^*_1, \ F''_1 = P_0 F''^*_1, \ F_{z_1} = L_0 P_0^2 F^*_{z_1}, \ F_{z_1 z_1} = P_0^2 F^*_{z_1 z_1}$$

and we insert them into (16). It then becomes apparent that the dimensionless quantities will also related to each other by equalities of the form (16), the only difference being that the multiplier $1/\mu$ is replaced σ.

If the modulus of the dimensionless functions is assumed to be finite and the parameter σ very small compared to unity, the terms containing σ in the dimensionless relations can be ignored because of their smallness relative to the other terms. We obtain an equivalent result if we ignore terms with the multiplier $1/\mu$ in the right sides of dimensionless relations (16). If we do so, these relations take the form

$$P_{xx} = -\mathrm{Re}\{(1 + 2\beta_1^2 - \beta_2^2)F''_1(z_1) + (1 + \beta_2^2)F''_2(z_2)\},$$
$$P_{yy} = (1 + \beta_2^2)\mathrm{Re}\{F''_1(z_1) + F''_2(z_2)\},$$
$$P_{xy} = \mathrm{Im}\,\{2\beta_1 F''_1(z_1) + \frac{(1 + \beta_2^2)^2}{2\beta_2} F''_2(z_2)\},\ \mu\omega_{xy} = \frac{1 - \beta_2^4}{4\beta_2}\mathrm{Im} F''_2(z_2),$$
$$\mu u_x = -\mathrm{Re}\{F'_1(z_1) + \frac{1 + \beta_2^2}{2} F'_2(z_2)\},\ \mu u_y = \mathrm{Im}\{\beta_1 F'_1(z_1) + \frac{1 + \beta_2^2}{2\beta_2} F'_2(z_2)\},$$

which agrees with Radok's solution [2] of dynamic equations of linear elasticity. Thus, the results of the linear theory follow from the results of the nonlinear theory for very small σ, i.e. for characteristic loads that are considerably below the elastic constant of the material.

To illustrate the use of Eqs. (16), we will examine the stress distribution in a semi-infinite two-dimensional elastic medium $y \geq 0$ due to the uniform motion of a pressure pulse along the boundary $y = 0$ at the velocity c. Having directed the x-axis along the boundary in the direction of the motion of the pulse, we take boundary conditions having the form

$$P_{xy} = 0,\ P_{yy} = -\mathrm{Re}\, H(x - ct) \text{ on } y = 0. \tag{17}$$

Assuming that $\eta = 0$ in the equation for P_{xy} in (16), we find that the first condition of (17) is satisfied if we take

$$F''_1(\xi) - \frac{1}{\mu} F_{\xi\xi} + \frac{(1 + \beta_2^2)^2}{4\beta_1\beta_2} F''_2(\xi) = 0.$$

The last equality will be satisfied if the following relationship exists between the arbitrary functions over the entire region:

$$F''_1(z_1) - \frac{1}{\mu} F_{z_1 z_1} = -\frac{(1 + \beta_2^2)^2}{4\beta_1\beta_2} F''_2(z_1).$$

Thus, stresses (16) are determined solely by the function F''_2:

$$\begin{aligned} P_{xx} = (1 + \beta_2^2)\Big\{(1 + 2\beta_1^2 - \beta_2^2) \frac{1 + \beta_2^2}{4\beta_1\beta_2} \mathrm{Re}\, F''_2(z_1) \\ - \mathrm{Re}\, F''_2(z_2)\Big\} + (1 - 2\beta_1^2 - \beta_2^2)\frac{\tau}{\mu}(\mathrm{Im}\, F''_2(z_2))^2, \\ P_{yy} = (1 + \beta_2^2)\Big\{\mathrm{Re}\, F''_2(z_2) - \frac{(1 + \beta_2^2)^2}{4\beta_1\beta_2}\mathrm{Re}\, F''_2(z_1) - \frac{\tau}{\mu}(\mathrm{Im}\, F''_2(z_2))^2\Big\}, \\ P_{xy} = \frac{(1 + \beta_2^2)^2}{2\beta_2}\Big\{\mathrm{Im}\, F''_2(z_2) - \mathrm{Im}\, F''_2(z_1)\Big\}. \end{aligned} \tag{18}$$

When Eq. (18) is used for the stress P_{yy}, the second condition of (17) leads to a nonlinear boundary-value problem for the analytic function F''_2:

$$\begin{gathered} m\mathrm{Re}\, F''_2(\xi) + n(\mathrm{Im}\, F''_2(\xi))^2 = \mathrm{Re}\, H(\xi), \\ m = (1 + \beta_2^2)\frac{(1 + \beta_2^2)^2 - 4\beta_1\beta_2}{4\beta_1\beta_2},\ n = (1 - \beta_2^4)\frac{\beta_1^2 - \beta_2^2}{2\mu}\left(\frac{1 + \beta_2^2}{4\beta_1\beta_2}\right)^2. \end{gathered} \tag{19}$$

In accordance with the above, the stress field and the boundary-value problem for the potential in linear elasticity follow from (18) and (19) if they do not contain any terms that include the multiplier $1/\mu$. In this case, boundary-value problem (19) becomes linear [2].

If we relate analytic function $F''_2(z_2)$ to complex function $\Phi(z_2, \bar{z}_2) = \Phi_1 + i\Phi_2$ by means of the expressions

$$\mathrm{Re}\, F''_2(z_2) = \frac{1}{m}\Phi_1(z_2,\bar{z}_2) - \frac{k}{m}\Phi_2^2(z_2,\bar{z}_2),\ \mathrm{Im}\, F''_2(z_2) = \frac{1}{m}\Phi_2(z_2,\bar{z}_2),$$
$$F''_2(z_2) = \frac{1}{m}\Phi(z_2,\bar{z}_2) + \frac{k}{4m}(\Phi(z_2,\bar{z}_2) - \overline{\Phi(z_2,\bar{z}_2)})^2,\ k = \frac{n}{m^2}, \tag{20}$$

then, as follows from the condition of analyticity of $F''_2(z_2)$ ($\partial_{\bar{z}_2}F''_2(z_2) = 0$) and boundary condition (19), this function satisfies the linear boundary-value problem for the quasilinear equation:

$$\frac{\partial\Phi}{\partial\bar{z}} - \gamma(\Phi,\overline{\Phi})\frac{\partial\overline{\Phi}}{\partial\bar{z}} = 0,\ \gamma = \frac{ik\Phi_2}{1 + ik\Phi_2},$$
$$\mathrm{Re}\,\Phi(\xi) = \mathrm{Re}\, H(\xi) \text{ on } \eta = 0. \tag{21}$$

It follows from (16) and (20) that the quantity Φ_2 entering into γ is proportional to the generalized rotation:

$$\Phi_2 = \frac{4\beta_2}{1 - \beta_2^4}\Omega_{xy}.$$

Since generalized rotations are assumed to be finite quantities in the theory being discussed, only the finite solutions of Eq. (21) have meaning in a mechanical sense. Since $|\gamma| < 1$ for these solutions, Eq. (21) is of the elliptic type [5]. Thus, if a finite solution is found for problem (21), then Eq. (20) determines the analytic function $F''_2(z_2)$. The latter in turn determines the stress field from Eqs. (18).

Let use examine an approximate solution of the nonlinear boundary-value problem for potential (19) that corresponds to a weak pressure pulse: $P_0 << \mu$ ($\sigma << 1$). Proceeding on the basis of the expressions

$$(F''_2)^2 = (\mathrm{Re}\, F''_2)^2 - (\mathrm{Im}\, F''_2)^2 + 2i\mathrm{Re}\, F''_2\mathrm{Im}\, F''_2,$$
$$(\mathrm{Im}\, F''_2)^2 = (\mathrm{Re}\, F''_2)^2 - \mathrm{Re}(F''_2)^2,$$

we represent problem (19) in the form

$$\mathrm{Re}\, F''_2(\xi) + \alpha[(\mathrm{Re}\, F''_2(\xi))^2 - \mathrm{Re}(F''_2(\xi))^2] = \mathrm{Re}\, h(\xi),\ \alpha = \frac{n}{m},\ h = \frac{H}{m}. \tag{22}$$

Written in dimensionless form, this equation contains the small dimensionless parameter

$$\alpha P_0 = \sigma\frac{\beta_1^2 - \beta_2^2}{2}\frac{1 + \beta_2^2}{4\beta_1\beta_2}\frac{1 - \beta_2^4}{(1 + \beta_2^2)^2 - 4\beta_1\beta_2},$$

which corresponds to the small dimensional parameter α. We represent the sought potential $F''_2(z_2)$ and the expressions in (22) that it determines in the form of expansions in the small parameter:

$$F''_2 = \varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 + \ldots,\ \mathrm{Re}\, F''_2 = \mathrm{Re}\,\varphi_0 + \alpha\mathrm{Re}\varphi_1 + \alpha^2\mathrm{Re}\varphi_2 + \ldots,$$
$$(\mathrm{Re}\, F''_2)^2 = (\mathrm{Re}\,\varphi_0)^2 + \alpha 2\mathrm{Re}\,\varphi_0\mathrm{Re}\,\varphi_1 + \alpha^2(2\mathrm{Re}\varphi_0\mathrm{Re}\varphi_2 + (\mathrm{Re}\varphi_1)^2) + \ldots, \tag{23}$$
$$\mathrm{Re}(F''_2)^2 = \mathrm{Re}(\varphi_0^2) + \alpha\mathrm{Re}(2\varphi_0\varphi_1) + \alpha^2\mathrm{Re}(2\varphi_0\varphi_2 + \varphi_1^2) + \ldots$$

If we substitute these expansions into (22) and equate the coefficients with identical powers of the parameter in the different parts of the equation, we arrive at the following system of equations for the functions φ_ν.

$$\mathrm{Re}\varphi_\nu(\xi) = w_\nu(\xi)\ (\nu = 0, 1, 2, \ldots) \text{ on } \eta = 0, \tag{24}$$

where

$$w_0 = \mathrm{Re}\, h;$$
$$w_1 = \mathrm{Re}(\varphi_0^2) - (\mathrm{Re}\,\varphi_0)^2 = -(\mathrm{Im}\,\varphi_0)^2;$$
$$w_2 = \mathrm{Re}(2\varphi_0\varphi_1) - 2\mathrm{Re}\,\varphi_0\mathrm{Re}\,\varphi_1 = -2\mathrm{Im}\varphi_0\mathrm{Im}\varphi_1;$$
$$\ldots\ldots$$

Here, the zeroth approximation corresponds to the analogous boundary-value problem of linear elasticity. In the equation for the ν-th approximation, the right part is determined by the previous approximations and is therefore known. Thus, in accordance with (24), each analytic function $\varphi_\nu(z_2)$ is expressed by the Schwarz formula for a half-plane [6] (with the assumption that the function $H(\xi)$ is bounded and at $|\xi| \to \infty$ approaches zero no more slowly than $1/|\xi|^e$, $e > 0$):

$$\varphi_\nu(z_2) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{w_\nu(\xi)}{\xi - z_2}\, d\xi \quad (\nu = 0, 1, 2, \ldots),$$

while the sought potential $F''_2(z_2)$ is expressed by series (23).

If we assign a rotation at the boundary in place of the second stress in (17), we can examine the boundary-value problem

$$P_{xy} = 0, \ \mu\omega_{xy} = \operatorname{Im} h(x - ct),$$

where $h(x - ct)$ is the boundary value of the analytic function $h(z_2)$. Then, as before, the first condition establishes the relationship between the arbitrary functions F''_1 and F''_2, which leads to Eqs. (18). These equations determine the stresses through a single function F''_2. By virtue of (16), the second boundary condition becomes the condition for the function F''_2:

$$\frac{1 - \beta_2^4}{4\beta_2} \operatorname{Im} F''_2(\xi) = \operatorname{Im} h(\xi) \ \text{on} \ y = 0.$$

This equation can be satisfied by taking the sought function proportional to $h(z_2)$ at all points of the region $y \geq 0$:

$$F''_2(z_2) = \frac{4\beta_2}{1 - \beta_2^4} h(z_2).$$

The resulting function F''_2 has the same form as when the given problem is solved within the framework of linear elasticity, since the rotation is expressed through F''_2 in the same manner in each case.

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